

# Bing doubling and the colored Jones polynomial

Sakie Suzuki\*

May 2, 2013

## Abstract

Bing doubling is an operation which gives a satellite of a knot. It is also applied to a link by specifying a component of the link. We give a formula to compute the reduced colored Jones polynomial of a Bing double by using that of the companion. This formula enables us to make a lot of examples of the reduced colored Jones polynomial of Bing doubles. Moreover, the result is applied to prove a divisibility property of the unified Witten-Reshetikhin-Turaev invariant of integral homology spheres obtained by  $\pm 1$ -surgery along Bing doubles in  $S^3$ .

## 1 Introduction

Bing doubling [1] is an operation which gives the satellite  $B(K)$  of a framed knot  $K$  as in Figure 1.

Bing doubling has been studied in the context of link concordance [2, 3, 4, 5, 7, 8, 9]. Bing doubling is also important in the study of Milnor's  $\bar{\mu}$  invariants [6] and finite type invariants of knots [11]. Some of classical invariants, for example, the multivariable Alexander polynomials or Arf invariants, are useless in the study of Bing doubles since those cannot distinguish iterated Bing doubles from unlink. Cimasoni [2] used Rasmussen invariant and Cha, Livingston and Ruberman [5] used the Ozváth-Szabó invariant and the Manolescu-Owens invariant to study Bing doubles.

Our interest here is in the relationship between Bing doubling and quantum invariants.

The colored Jones polynomial  $J_{L;W_1,\dots,W_n} \in \mathbb{Z}[q^{1/4}, q^{-1/4}]$  is one of quantum invariants, which is defined for a framed link  $L = L_1 \cup \dots \cup L_n$  with the  $i$ th component colored

---

\*Faculty of Mathematics, Kyushu University, Fukuoka, 819-0395, Japan. E-mail address: sakie@math.kyushu-u.ac.jp

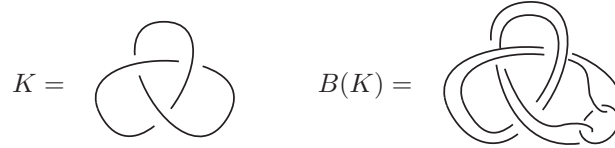


Figure 1: The trefoil knot  $K$  and its Bing double  $B(K)$

by a finite dimensional representation  $W_i$  of the quantized enveloping algebra  $U_h(sl_2)$  of the Lie algebra  $sl_2$ .

Habiro [10] defined *the reduced colored Jones polynomial*  $J_{L;P'_1,\dots,P'_{l_n}} \in \mathbb{Q}(q^{\frac{1}{2}})$ , where  $P'_{l_i}$ ,  $l_i \geq 0$ , is an element in the representation ring of  $U_h(sl_2)$  over  $\mathbb{Q}(q^{\frac{1}{2}})$ . By using the reduced colored Jones polynomial, he constructed *the unified Witten-Reshetikhin-Turaev (WRT) invariant*  $J_M \in \widehat{\mathbb{Z}[q]}$  of an integral homology sphere  $M$ , where  $\widehat{\mathbb{Z}[q]}$  is so-called *the Habiro ring*. See Sections 2.1 and 2.2 for the definitions of the reduced colored Jones polynomial and the WRT invariant.

For a framed link  $L = L_1 \cup \dots \cup L_n$ , we denote by  $B(L; 1)$  the  $(n+1)$ -component link obtained from  $L$  by applying Bing doubling to  $L_1$ . It is not difficult to see that the reduced colored Jones polynomial of  $B(L; 1)$  is a linear combination of that of  $L$  (Lemma 3.1), i.e., we have

$$J_{B(L;1);P'_i,P'_j,P'_{l_2},\dots,P'_{l_n}} = \sum_{l \geq 0} x_{i,j}^{(l)} J_{L;P'_l,P'_{l_2},\dots,P'_{l_n}}$$

for certain  $x_{i,j}^{(l)} \in \mathbb{Q}(q^{\frac{1}{2}})$ .

One of two main results in this paper is Theorem 3.2, which gives the coefficients  $x_{i,j}^{(l)}$  explicitly. This result gives many examples of computations of the reduced colored Jones polynomials of Bing doubles.

The other one is Theorem 3.3, which says a divisibility property of the difference  $J_{M(B(K);\epsilon,\epsilon')} - J_M$  of unified WRT invariants, where  $M$  is an integral homology sphere and  $K$  is a 0-framed knot in  $M$ , and  $M(B(K);\epsilon,\epsilon')$  is the integral homology sphere obtained from  $M$  by surgery along the Bing double  $B(K)$  with  $\epsilon, \epsilon' \in \{\pm 1\}$  framing.

The rest of the paper is organized as follows. In Section 2, we recall the definitions of the reduced colored Jones polynomial and the unified WRT invariant. In Section 3, we give the main results. In Section 4, we study the coefficients  $x_{i,j}^{(l)} \in \mathbb{Q}(q^{\frac{1}{2}})$  further. In Section 5, we compute some of the reduced colored Jones polynomials of Milnor's link as in Figure 4, which is obtained from the Borromean rings by applying Bing doubling repeatedly. Section 6 is devoted to the proofs.

## 2 Preliminaries

In this section, we recall from [10] the definition of the reduced colored Jones polynomial and the unified WRT invariant.

We use the following  $q$ -integer notations:

$$\begin{aligned} \{i\} &= q^{i/2} - q^{-i/2}, & \{i\}_n &= \{i\}\{i-1\}\cdots\{i-n+1\}, \\ \{n\}! &= \{n\}_n, & \begin{bmatrix} i \\ n \end{bmatrix} &= \{i\}_n / \{n\}!, \end{aligned}$$

for  $i \in \mathbb{Z}, n \geq 0$ .

## 2.1 Reduced Colored Jones polynomial

The colored Jones polynomial  $J_{L;W_1,\dots,W_n} \in \mathbb{Z}[q^{1/4}, q^{-1/4}]$  is defined for an  $n$ -component framed link  $L$  with the  $i$ th component  $L_i$  colored by a finite dimensional representation  $W_i$  of the quantized enveloping algebra  $U_h(sl_2)$  of the Lie algebra  $sl_2$ . In this paper, we follow [10] for the definition of colored Jones polynomial. The reduced colored Jones polynomial is a linear combination of the colored Jones polynomial as follows.

For  $m \geq 0$ , let  $V_m$  denote the  $(m+1)$ -dimensional irreducible representation of  $U_h(sl_2)$ . Let  $\mathcal{R}$  denote the representation ring of  $U_h(sl_2)$  over  $\mathbb{Q}(q^{\frac{1}{2}})$ , i.e.,  $\mathcal{R}$  is the  $\mathbb{Q}(q^{\frac{1}{2}})$ -algebra

$$\mathcal{R} = \text{Span}_{\mathbb{Q}(q^{\frac{1}{2}})} \{V_m \mid m \geq 0\}$$

with the multiplication induced by the tensor product.

We define the colored Jones polynomial of an  $n$ -component framed link  $L$  with the  $i$ th component  $L_i$  colored by  $X_i = \sum_{l_i \geq 1} a_{l_i}^{(i)} V_{l_i} \in \mathcal{R}$  by

$$J_{L;X_1,\dots,X_n} = \sum_{l_1,\dots,l_n \geq 1} a_{l_1}^{(1)} \cdots a_{l_n}^{(n)} J_{L;V_{l_1},\dots,V_{l_n}} \in \mathbb{Q}(q^{\frac{1}{2}}).$$

For  $l \geq 0$ , set

$$P'_l = \frac{1}{\{l\}!} \prod_{i=0}^{l-1} (V_1 - q^{i+\frac{1}{2}} - q^{-i-\frac{1}{2}}) \in \mathcal{R}.$$

For an  $n$ -component framed link  $L$ , we call  $J_{L;P'_{l_1},\dots,P'_{l_n}}$  the *reduced colored Jones polynomial* of  $L$ .

## 2.2 Unified WRT invariant

For  $k \geq 0$ , set

$$\mathcal{P}_k = \text{Span}_{\mathbb{Z}[q,q^{-1}]} \{q^{-\frac{1}{4}l(l-1)} P'_l \mid l \geq k\} \subset \mathcal{R},$$

Set

$$\hat{\mathcal{P}} = \varprojlim_{k \geq 0} \mathcal{P}_0 / \mathcal{P}_k.$$

Set

$$\omega^{\pm 1} = \sum_{l=0}^{\infty} (\pm 1)^l q^{\pm \frac{1}{4}l(l+3)} P'_l \in \hat{\mathcal{P}}.$$

Let  $\widehat{\mathbb{Z}[q]}$  be the Habiro ring, i.e.,

$$\widehat{\mathbb{Z}[q]} = \varprojlim_{n \geq 0} \mathbb{Z}[q] / ((1-q)(1-q^2) \cdots (1-q^n)).$$

Let  $M$  be the integral homology sphere obtained by surgery along an algebraically-split link  $L = L_1 \cup \dots \cup L_n$  in  $S^3$  with framings  $\epsilon_1, \dots, \epsilon_n \in \{\pm 1\}$ . Habiro [10] constructed the unified WRT invariant  $J_M \in \widehat{\mathbb{Z}[q]}$  of  $M$  by

$$J_M = J_{L^0; \omega^{-\epsilon_1}, \dots, \omega^{-\epsilon_n}} \\ := \sum_{l_1, \dots, l_n=0}^{\infty} \left( \prod_{i=1, \dots, n} (-\epsilon_i)^{l_i} q^{-\epsilon_i \frac{1}{4} l_i (l_i+3)} \right) J_{L^0; P'_{l_1}, \dots, P'_{l_n}} \in \widehat{\mathbb{Z}[q]},$$

where  $L^0$  is the link obtained from  $L$  by changing all framings to 0.

### 3 Main results

In this section, we give main results for the reduced colored Jones polynomial (Theorem 3.2) and for the unified WRT invariant (Theorem 6.5).

#### 3.1 Result for the reduced colored Jones polynomial

We give the result for the reduced colored Jones polynomial.

It is known that the colored Jones polynomial of a satellite is a linear combination of the colored Jones polynomials of the companion [14, 12]. The following lemma says that a similar situation works for the reduced colored Jones polynomials of Bing doubles.

**Lemma 3.1.** *There exists  $x_{i,j}^{(l)} \in \mathbb{Q}(q^{1/2})$ ,  $i, j, l \geq 0$ , such that*

$$J_{B(L;1); P'_i, P'_j, W_1, \dots, W_{n-1}} = \sum_{l \geq 0} x_{i,j}^{(l)} J_{L; P'_l, W_1, \dots, W_{n-1}},$$

for any  $n$ -component link  $L = L_1 \cup \dots \cup L_n$  with 0-framing and  $W_1, \dots, W_{n-1} \in \mathcal{R}$ .

*Proof.* For simplicity, we prove the claim for  $n = 1$ , i.e.,  $L = K$  is a knot with 0-framing. We can prove the assertion for  $n \geq 2$  similarly. Since  $\{V_l\}_{l \geq 0}$  is a basis of  $\mathcal{R}$ , the colored Jones polynomial  $J_{B(K); P'_i, P'_j}$  is a linear sum of colored Jones polynomials  $J_{B(K); V_t, V_u}$  in  $\mathcal{R}$ . Since the colored Jones polynomial  $J_{B(K); V_t, V_u}$  is a linear sum of colored Jones polynomials  $J_{K; V_s}$  in  $\mathbb{Z}[q^{1/2}, q^{-1/2}]$  (cf. [12, Theorem 3.1]), and since  $\{P'_l\}_{l \geq 0}$  is also a basis of  $\mathcal{R}$ ,  $J_{B(K); V_t, V_u}$  is a linear sum of colored Jones polynomials  $J_{K; P'_l}$  in  $\mathcal{R}$ . Consequently,  $J_{B(K); P'_i, P'_j}$  is a linear sum of  $J_{K; P'_l}$  in  $\mathcal{R}$ . Moreover, in each step, the coefficients of the linear sum do not depend on the knot  $K$ . Hence we have the assertion.  $\square$

The main result for the reduced colored Jones polynomial in this paper is the following, which we prove in Section 6.1.

**Theorem 3.2.** *For  $i, j, l \geq 0$ , we have*

$$x_{i,j}^{(l)} = \delta_{i,j} (-1)^i \{l\}! \lambda_{i,l}, \quad (1)$$

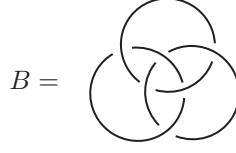


Figure 2: Borromean rings  $B$

where

$$\lambda_{i,l} = \sum_{k=0}^i (-1)^k \begin{bmatrix} 2i+1 \\ k \end{bmatrix} \begin{bmatrix} 2i+l-2k+1 \\ 2l+1 \end{bmatrix}.$$

Theorem 3.2 enables us to compute many examples of the reduced colored Jones polynomial of Bing doubles. Especially in Section 5, we give examples with Milnor's links.

Moreover, by using Theorem 3.2, we can prove a divisibility property of the unified WRT invariant as in the following section.

### 3.2 Result for the unified WRT invariant

For  $m \geq 1$ , let  $\Phi_m = \prod_{d|m} (q^d - 1)^{\mu(\frac{m}{d})} \in \mathbb{Z}[q]$  denote the  $m$ th cyclotomic polynomial, where  $\prod_{d|m}$  denotes the product over all the positive divisors  $d$  of  $m$ , and  $\mu$  is the Möbius function.

Let  $M$  be an integral homology sphere, and  $L = L_1 \cup \dots \cup L_n$  an algebraically-split link in  $M$ . For  $\epsilon_1, \dots, \epsilon_n \in \{\pm 1\}$ , let  $M(L; \epsilon_1, \dots, \epsilon_n)$  denote the integral homology sphere obtained from  $M$  by surgery along  $L$  with framings  $\epsilon_1, \dots, \epsilon_n$ . The result for the unified WRT invariant is the following, which we prove in Section 6.2.

**Theorem 3.3.** *Let  $M$  be an integral homology sphere, and  $K$  a knot in  $M$ . For  $\epsilon, \epsilon' \in \{\pm 1\}$ , we have*

$$J_{M(B(K); \epsilon, \epsilon')} - J_M \in \Phi_1^2 \Phi_2^2 \Phi_3 \Phi_4 \Phi_6 \widehat{\mathbb{Z}[q]} = (q^4 - 1)(q^6 - 1) \widehat{\mathbb{Z}[q]}.$$

In particular, we have the following.

**Corollary 3.4.** *For a knot  $K$  in  $S^3$  and  $\epsilon, \epsilon' \in \{\pm 1\}$ , we have*

$$J_{S^3(B(K); \epsilon, \epsilon')} - 1 \in \Phi_1^2 \Phi_2^2 \Phi_3 \Phi_4 \Phi_6 \widehat{\mathbb{Z}[q]}.$$

**Remark 3.5.** Note that  $S^3(B(K); \epsilon, \epsilon') = S^3(W_{\epsilon'}(K), \epsilon)$ , where  $W_{\epsilon'}(K)$  is the Whitehead double of  $K$  with a clasp of  $\epsilon'$ -type.

**Remark 3.6.** Habiro [10, Proposition 12.15] proved that for an integral homology sphere  $M$  with vanishing Casson invariant,  $J_M - 1$  is divisible by  $\Phi_1^2 \Phi_2^2 \Phi_3 \Phi_4 \Phi_6$ . Since  $M(B(K); \epsilon, \epsilon')$  in Corollary 3.4 has vanishing Casson invariant, Corollary 3.4 is a refinement of Habiro's result with respect to  $\Phi_2$ .

There is a family of examples of integral homology spheres which do not have the divisibility property in Corollary 3.4 as follows.

Let  $B$  be the Borromean rings depicted in Figure 2 and  $B_1, B_2, B_3$  the components of  $B$ . For  $i, j, k \in \mathbb{Z}$ , let  $M_{i,j,k}$  be the integral homology sphere obtained from  $S^3$  by surgery along  $B_1, B_2, B_3$  with framings  $-1/i, -1/j, -1/k$ , respectively.

We prove the following proposition in Section 6.3.

**Proposition 3.7.** *For  $i, j, k \in \mathbb{Z}$ , we have*

$$J_{M_{i,j,k}} - 1 \equiv 6ijk\Phi_2 \pmod{\Phi_2^2}.$$

Since the Casson invariant of  $M_{i,j,k}$  is  $6ijk$  up to the signature, we have the following conjecture.

**Conjecture 3.8.** *Let  $M$  be an integral homology sphere with vanishing Casson invariant. Then we have*

$$J_M - 1 \in \Phi_1^2 \Phi_2^2 \Phi_3 \Phi_4 \widehat{\Phi_6 \mathbb{Z}[q]}.$$

## 4 Properties of the coefficients $x_{i,j}^{(l)}$ in Theorem 3.2.

In this section, we study the coefficients  $x_{i,j}^{(l)} = \delta_{i,j}(-1)^i \{l\}! \lambda_{i,l}$  which appears in Theorem 3.2.

### 4.1 Symmetry property

For  $m, n \geq 0$ , recall

$$\lambda_{m,n} = \sum_{k=0}^{\lfloor m - \frac{n}{2} \rfloor} (-1)^k \begin{bmatrix} 2m+1 \\ k \end{bmatrix} \begin{bmatrix} 2m+n-2k+1 \\ 2n+1 \end{bmatrix}, \quad (2)$$

where the upper bound comes from

$$\begin{bmatrix} 2m+n-2k+1 \\ 2n+1 \end{bmatrix} = 0$$

for  $2m+n-2k+1 < 2n+1$ , i.e., for  $k > m - n/2$ .

The element  $\lambda_{m,n}$  has a symmetry property as follows, which we prove in Section 6.1

**Lemma 4.1.** *For  $m, n \geq 0$ , we have*

$$\lambda_{m,n} = \frac{\{2m+1\}!}{\{2n+1\}!} \lambda_{n,m}.$$

**Corollary 4.2.** *For  $m, n \geq 0$ , we have*

$$x_{m,m}^{(n)} = (-1)^{m+n} \frac{\{2m+1\}! \{n\}!}{\{2n+1\}! \{m\}!} x_{n,n}^{(m)}.$$

*Proof.* By (1) and Lemma 4.1, we have

$$\begin{aligned}
x_{m,m}^{(n)} &= (-1)^m \{n\}! \lambda_{m,n} \\
&= (-1)^m \{n\}! \frac{\{2m+1\}!}{\{2n+1\}!} \lambda_{n,m} \\
&= (-1)^m \{n\}! \frac{\{2m+1\}!}{\{2n+1\}!} \left( (-1)^n \frac{1}{\{m\}!} x_{n,n}^{(m)} \right) \\
&= (-1)^{m+n} \frac{\{2m+1\}! \{n\}!}{\{2n+1\}! \{m\}!} x_{n,n}^{(m)}.
\end{aligned}$$

Hence we have the assertion.  $\square$

## 4.2 Particular values

We can compute particular values of  $\lambda_{m,n}$  as follows.

**Proposition 4.3.** *We have*

- (i)  $\lambda_{m,n} = 0$  unless  $\frac{n}{2} \leq m \leq 2n$ ,
- (ii)  $\lambda_{m,2m} = 1$  for  $m \geq 0$ , and
- (iii)  $\lambda_{m,2m-1} = \{4m\}/\{1\}$  for  $m \geq 0$ .

*Proof.* The assertion (i) follows from (2) and Lemma 4.1.

The assertion (ii) follows from

$$\begin{aligned}
\lambda_{m,2m} &= \sum_{k=0}^0 (-1)^k \begin{bmatrix} 2m+1 \\ k \end{bmatrix} \begin{bmatrix} 4m-2k+1 \\ 4m+1 \end{bmatrix} \\
&= \begin{bmatrix} 2m+1 \\ 0 \end{bmatrix} \begin{bmatrix} 4m+1 \\ 4m+1 \end{bmatrix} \\
&= 1.
\end{aligned}$$

The assertion (iii) follows from

$$\begin{aligned}
\lambda_{m,2m-1} &= \sum_{k=0}^0 (-1)^k \begin{bmatrix} 2m+1 \\ k \end{bmatrix} \begin{bmatrix} 4m-2k \\ 4m-1 \end{bmatrix} \\
&= \begin{bmatrix} 2m+1 \\ 0 \end{bmatrix} \begin{bmatrix} 4m \\ 4m-1 \end{bmatrix} \\
&= \begin{bmatrix} 4m \\ 1 \end{bmatrix} \\
&= \{4m\}/\{1\}.
\end{aligned}$$

This completes the proof.  $\square$

Proposition 4.3 implies the following.

**Corollary 4.4.** *We have*

- (i)  $x_{m,m}^{(n)} = 0$  unless  $\frac{n}{2} \leq m \leq 2n$ ,
- (ii)  $x_{m,m}^{(2m)} = (-1)^m \{2m\}!$  for  $m \geq 0$ , and
- (iii)  $x_{m,m}^{(2m-1)} = (-1)^m \{2m-1\}! \{4m\} / \{1\}$  for  $m \geq 0$ .

### 4.3 Divisibility property with respect to the cyclotomic polynomials

Let us study the divisibility property of  $\lambda_{m,n}$  and  $x_{n,n}^{(m)}$  with respect to the cyclotomic polynomials. In what follows, we use also the symmetric version of the cyclotomic polynomial  $\tilde{\Phi}_l = \prod_{d|l} (q^{d/2} - q^{-d/2})^{\mu(\frac{l}{d})} \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$  for  $l \geq 1$ . For  $f \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$ ,  $f \neq 0$ , let  $d_l(f)$  be the largest integer  $i$  such that  $f \in \tilde{\Phi}_l^i \mathbb{Z}[q^{1/2}, q^{-1/2}]$ .

**Lemma 4.5.** *For  $l \geq 1$  and  $n \geq 0$ , we have*

$$d_l(\{n\}!) = \lfloor \frac{n}{l} \rfloor.$$

*Proof.* The assertion follows from

$$d_l(\{i\}) = d_l(q^{i/2} - q^{-i/2}) = \begin{cases} 1 & \text{if } l|i, \\ 0 & \text{otherwise,} \end{cases}$$

for  $i \geq 0$ . □

Lemma 4.5 implies the following result.

**Corollary 4.6.** *For  $l \geq 1$  and  $m, n \geq 0$ , we have*

$$d_l(x_{m,m}^{(n)}) = \lfloor \frac{n}{l} \rfloor + d_l(\lambda_{m,n}).$$

In what follows, we study  $d_l(\lambda_{m,n})$ . Lemmas 4.1 and 4.5 imply the following result.

**Corollary 4.7.** *For  $l \geq 1$  and  $m, n \geq 0$ , we have*

$$d_l(\lambda_{m,n}) = \lfloor \frac{2m+1}{l} \rfloor - \lfloor \frac{2n+1}{l} \rfloor + d_l(\lambda_{n,m}).$$

With respect to  $\tilde{\Phi}_1$ , we have the following result.

**Proposition 4.8.** *For  $m, n \geq 0$ , we have*

$$d_1(\lambda_{m,n}) = \begin{cases} 2(m-n) & \text{if } 0 \leq n \leq m < 2n, \\ 0 & \text{if } 0 \leq m \leq n \leq 2m. \end{cases}$$

We prove Proposition 4.8 by using the following lemma.



**Lemma 4.9.** For  $0 \leq j \leq m$ , we have

$$\lambda_{m,2m-j}|_{q^{1/2}=1} = 4^j \binom{m}{j}.$$

*Proof of Proposition 4.8.* By Lemma 4.9, for  $0 \leq m \leq n \leq 2m$ , we have  $\lambda_{m,n} \neq 0$  and  $\lambda_{m,n}|_{q^{1/2}=1} \neq 0$ , which implies  $d_1(\lambda_{m,n}) = 0$ . The assertion for  $0 \leq n \leq m < 2n$  follows from  $d_1(\lambda_{n,m}) = 0$  and Corollary 4.7.  $\square$

*Proof of Lemma 4.9.* Set  $\tilde{\lambda}_{m,j} = \lambda_{m,2m-j}|_{q^{1/2}=1}$ . It is enough to prove

$$j\tilde{\lambda}_{m,j} - 4(m-j+1)\tilde{\lambda}_{m,j-1} = 0, \quad (3)$$

which implies

$$\begin{aligned} \tilde{\lambda}_{m,j} &= 4 \frac{(m-j+1)}{j} \tilde{\lambda}_{m,j-1} \\ &= \dots = 4^j \frac{(m-j+1)(m-j+2)\cdots(m)}{j(j-1)\cdots 1} \tilde{\lambda}_{m,0} = 4^j \binom{m}{j}, \end{aligned}$$

where  $\tilde{\lambda}_{m,0} = 1$  by Proposition 4.3 (ii).

We prove (3). Note that

$$\tilde{\lambda}_{m,j} = \lambda_{m,2m-j}|_{q^{1/2}=1} = \sum_{k=0}^{\lfloor m - \frac{2m-j}{2} \rfloor} F(m, j, k) = \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \tilde{F}(m, j, k),$$

with

$$\begin{aligned} F(m, n, k) &= (-1)^k \binom{2m+1}{k} \binom{2m+n-2k+1}{2n+1}, \\ \tilde{F}(m, j, k) &= F(m, 2m-j, k). \end{aligned}$$

If we can find  $G(m, j, k) \in \mathbb{Q}$  for  $0 \leq k \leq \lfloor \frac{j}{2} \rfloor + 1$  such that

$$j\tilde{F}(m, j, k) - 4(m-j+1)\tilde{F}(m, j-1, k) = G(m, j, k+1) - G(m, j, k) \quad (4)$$

and

$$G(m, j, 0) = G(m, j, \lfloor \frac{j}{2} \rfloor + 1) = 0, \quad (5)$$

then (3) follows from

$$j\tilde{\lambda}_{m,j} - 4(m-j+1)\tilde{\lambda}_{m,j-1} = G(m, j, \lfloor \frac{j}{2} \rfloor + 1) - G(m, j, 0) = 0.$$

Actually we can find such  $G(m, j, k)$  by using Zeilberger's algorithm [13] as follows.

$$G(m, j, k) = \begin{cases} -\frac{2k(2k-4m+j-3)(2k-4m+j-2)}{(4m-2j+2)(4m-2j+3)} \tilde{F}(m, j, k) & \text{for } 1 \leq k \leq \lfloor \frac{j}{2} \rfloor, \\ 0 & \text{for } k = 0, \lfloor \frac{j}{2} \rfloor + 1. \end{cases}$$

We can check (4) and (5) by straightforward calculations. Hence we have the assertion.  $\square$

$m \setminus n$	1	2	3	4	5	6	7	8	9	10
1	0	0								
2	2	0	0	0						
3		2	0	0	0	0				
4		4	2	0	0	0	0	0		
5			4	2	0	0	0	0	0	0
6			6	4	2	0	0	0	0	0
7				6	4	2	0	0	0	0
8				8	6	4	2	0	0	0

Table 1:  $d_1(\lambda_{m,n})$

$m \setminus n$	1	2	3	4	5	6	7	8	9	10
1	1	0								
2	1	0	1	0						
3		2	1	0	1	0				
4		2	1	0	1	0	1	0		
5			3	2	1	0	1	0	1	0
6			3	2	1	0	1	0	1	0
7				4	3	2	1	0	1	0
8				4	3	2	1	0	1	0

Table 2:  $d_2(\lambda_{m,n})$

See Table 1 for the behavior of  $d_1(\lambda_{m,n})$ , where we color the boxes gray for  $0 < n < m$  and the blanks mean  $\lambda_{m,n} = 0$ .

Though we do not have a general result for  $d_l(\lambda_{m,n})$  for  $l \geq 2$ , we can compute that for small  $m, n \geq 0$ . See Table 2–5 for  $d_l(\lambda_{m,n})$  for  $l = 2, \dots, 5$ .

We have the following conjecture, which enables us to compute  $d_l(\lambda_{m,n})$  for general  $m, n \geq 0$  by using explicit computation of  $d_l(\lambda_{m,n})$  for small  $m, n \geq 0$ .

**Conjecture 4.10.** *For  $l \geq 1$ , in the range  $0 \leq m \leq n \leq 2m$ ,  $d_l(\lambda_{m,n})$  is periodic with period  $l$  both in  $m$  and in  $n$ , i.e., for  $m \equiv \tilde{m} \pmod{l}$ , and  $n \equiv \tilde{n} \pmod{l}$ , we have*

$$d_l(\lambda_{m,n}) = d_l(\lambda_{\tilde{m},\tilde{n}}).$$

We also have the following conjecture.

**Conjecture 4.11.** *For a prime  $l \geq 1$  and  $0 \leq m \leq n \leq 2m$ , we have  $d_l(\lambda_{m,n}) \in \{0, 1\}$ .*

## 5 Colored Jones polynomial of Milnor's link

In this section, we give examples of computations of the reduced colored Jones polynomials of Milnor's link.

$m \setminus n$	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0	0											
2	0	0	0	0									
3		1	0	1	1	0							
4		2	2	0	0	1	0	0					
5			2	0	0	0	0	0	0	0			
6			2	2	1	0	1	1	0	1	1	0	
7				2	2	2	0	0	1	0	0	1	0
8				2	2	2	0	0	0	0	0	0	0

Table 3:  $d_3(\lambda_{m,n})$

$m \setminus n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0													
2	1	0	1	0											
3		1	1	0	1	0									
4		1	1	0	2	1	1	0							
5			2	2	1	0	2	1	1	0					
6			2	2	1	0	1	0	1	0	1	0			
7				2	3	1	1	0	1	0	1	0	1	0	
8				2	3	1	1	0	2	1	1	0	2	1	1

Table 4:  $d_4(\lambda_{m,n})$

$m \setminus n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	0													
2	1	0	0	0											
3		0	0	0	0	0									
4		0	0	0	0	0	0	0							
5			1	1	0	1	1	1	1	0					
6			1	1	1	0	0	1	1	1	0	0			
7				2	2	1	0	0	0	1	1	0	0	0	
8				2	2	2	0	0	0	0	0	0	0	0	0
9					2	2	0	0	0	0	0	0	0	0	0
10					2	3	2	1	1	0	1	1	1	1	0

Table 5:  $d_5(\lambda_{m,n})$

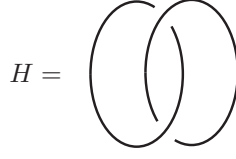


Figure 3: The Hopf link  $H$



Figure 4: Milnor's link  $A_n$

For  $n \geq 3$ , let  $A_n$  be the  $n$ -component Milnor's link depicted in Figure 4. Let  $A_2 = H$  be the Hopf link depicted in Figure 3. Note that  $A_3 = B$  is the Borromean rings, and  $A_n = B(A_{n-1}; 1)$  for  $n \geq 3$ .

We use the following result.

**Lemma 5.1** (Habiro [10, Corollary 14.2]). *For  $i, j, k \geq 0$ , we have*

$$J_{B; P'_i, P'_j, P'_k} = \begin{cases} (-1)^i \{2i+1\}_{i+1} / \{1\} & \text{if } i = j = k, \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 3.2 and Lemma 5.1, we can compute the reduced colored Jones polynomial of  $A_n$  as follows.

**Proposition 5.2.** *For  $n \geq 3$ , we have*

$$\begin{aligned} J_{A_n; P'_{l_1}, \dots, P'_{l_n}} &= \delta_{l_1, l_2} \delta_{l_{n-1}, l_n} x_{l_2, l_2}^{(l_3)} x_{l_3, l_3}^{(l_4)} \cdots x_{l_{n-2}, l_{n-2}}^{(l_{n-1})} J_{B; P'_{l_{n-1}}, P'_{l_{n-1}}, P'_{l_{n-1}}} \\ &= \delta_{l_1, l_2} \delta_{l_{n-1}, l_n} x_{l_2, l_2}^{(l_3)} x_{l_3, l_3}^{(l_4)} \cdots x_{l_{n-2}, l_{n-2}}^{(l_{n-1})} (-1)^{l_{n-1}} \{2l_{n-1} + 1\}_{l_{n-1}+1} / \{1\}. \end{aligned}$$

*Proof.* We use induction on  $n$ . For  $n = 3$ , by Lemma 5.1, we have

$$J_{B; P'_{l_1}, P'_{l_2}, P'_{l_3}} = \delta_{l_1, l_2} \delta_{l_2, l_3} J_{B; P'_{l_2}, P'_{l_2}, P'_{l_2}}.$$

For  $n > 3$ , by the assumption of induction, we have

$$\begin{aligned} J_{A_n; P'_{l_1}, \dots, P'_{l_n}} &= \sum_{k_1 \geq 0} x_{l_1, l_2}^{(k_1)} J_{A_{n-1}; P'_{k_1}, P'_{l_3}, \dots, P'_{l_n}} \\ &= \delta_{l_1, l_2} \sum_{k_1 \geq 0} x_{l_2, l_2}^{(k_1)} J_{A_{n-1}; P'_{k_1}, P'_{l_3}, \dots, P'_{l_n}} \\ &= \delta_{l_1, l_2} \sum_{k_1 \geq 0} x_{l_2, l_2}^{(k_1)} \delta_{k_1, l_3} \delta_{l_{n-1}, l_n} x_{l_3, l_3}^{(l_4)} x_{l_4, l_4}^{(l_5)} \cdots x_{l_{n-2}, l_{n-2}}^{(l_{n-1})} J_{A; P'_{l_{n-1}}, P'_{l_{n-1}}, P'_{l_{n-1}}} \\ &= \delta_{l_1, l_2} \delta_{l_{n-1}, l_n} x_{l_1, l_2}^{(l_3)} x_{l_3, l_3}^{(l_4)} x_{l_4, l_4}^{(l_5)} \cdots x_{l_{n-2}, l_{n-2}}^{(l_{n-1})} J_{A; P'_{l_{n-1}}, P'_{l_{n-1}}, P'_{l_{n-1}}}. \end{aligned}$$

Thus we have the assertion.  $\square$

We have the following corollaries.

**Corollary 5.3.** *For  $n \geq 3$ , we have*

$$J_{A_n; P'_1, \dots, P'_1} = (-1)^n \tilde{\Phi}_1^{n-2} \tilde{\Phi}_2^{n-2} \tilde{\Phi}_3 \tilde{\Phi}_4^{n-3}.$$

*Proof.* The assertion follows from Proposition 5.2 and

$$x_{1,1}^{(1)} = -\{4\} = -\tilde{\Phi}_4 \tilde{\Phi}_2 \tilde{\Phi}_1 \quad (6)$$

and

$$J_{B; P'_1, P'_1, P'_1} = -\{3\}_2 / \{1\} = -\tilde{\Phi}_3 \tilde{\Phi}_2 \tilde{\Phi}_1. \quad (7)$$

□

**Corollary 5.4.** (i) *For  $n \geq 3$  and  $l_1, \dots, l_n \geq 0$ , unless  $l_1 = l_2$ ,  $l_{n-1} = l_n$  and unless  $\frac{1}{2} \leq \frac{l_{i+1}}{l_i} \leq 2$  for  $2 \leq i \leq n-2$ , we have  $J_{A_n; P'_{l_1}, \dots, P'_{l_n}} = 0$ .*

(ii) *For  $a_0 \geq 0$  and  $a_i = 2a_{i-1}$  for  $i = 1, \dots, n-3$ , we have*

$$J_{A_n; P'_{a_0}, P'_{a_0}, P'_{a_1}, \dots, P'_{a_{n-4}}, P'_{a_{n-3}}, P'_{a_{n-3}}} = \left( \prod_{m=1}^{n-3} (-1)^{a_m} \{2a_m\}! \right) ((-1)^{a_{n-3}} \{2a_{n-3} + 1\}_{a_{n-3}+1} / \{1\}).$$

(iii) *For  $b_0 \geq 0$  and  $b_i = 2b_{i-1} - 1$  for  $i = 1, \dots, n-3$  (i.e.,  $b_i = 2^i b_0 - \frac{(i)(i+1)(2i+1)}{6}$ ) we have*

$$J_{A_n; P'_{b_0}, P'_{b_0}, P'_{b_1}, \dots, P'_{b_{n-4}}, P'_{b_{n-3}}, P'_{b_{n-3}}} = \left( \prod_{m=1}^{n-3} (-1)^{b_m} \{2b_m - 1\}! \{4b_m\} / \{1\} \right) ((-1)^{b_{n-3}} \{2b_{n-3} + 1\}_{b_{n-3}+1} / \{1\}).$$

*Proof.* The assertions (i), (ii) and (iii) follow from Corollary 4.4 (i), (ii) and (iii), respectively, and Proposition 5.2.

□

## 6 Proofs

In this section, we prove Theorems 3.2, 6.5, and Proposition 3.7.

### 6.1 Proof of Theorem 3.2

We prove Theorem 3.2. We also prove Lemma 4.1 at the end of this section.

In [10], Habiro defined the element

$$S_n = \prod_{i=1}^n (V_2 - (q^i + 1 + q^{-i})) \in \mathcal{R},$$

for  $n \geq 0$ , which is a kind of dual of  $P'_n$  with respect to the symmetric bilinear form  $J_{H; -, -} : \mathcal{R} \times \mathcal{R} \rightarrow \mathbb{Q}(q^{1/2})$  as follows.

**Lemma 6.1** (Habiro [10, Proposition 6.6]). *For  $m, n \geq 0$ , we have*

$$J_{H;P'_m,S_n} = \delta_{m,n} \{2m+1\}_{2m}/\{m\}!.$$

Recall the element  $\lambda_{m,n} = \lambda_{m,n}(q^{1/2}) \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$  defined in Theorem 3.2 (ii). We reduce Theorem 3.2 to the following proposition.

**Proposition 6.2.** *For  $l \geq 0$ , we have*

$$S_l = \sum_{m \geq 0} \lambda_{l,m} \{m\}! P'_m.$$

*Proof of Theorem 3.2 assuming Proposition 6.2.* By Proposition 6.2 and Lemma 5.1, we have

$$\begin{aligned} J_{B;P'_i,P'_j,S_l} &= \sum_{m \geq 0} \lambda_{l,m} \{m\}! J_{B;P'_i,P'_j,P'_m} \\ &= \delta_{i,j} \lambda_{l,i} \{i\}! (-1)^i \{2i+1\}_{i+1}/\{1\} \\ &= \delta_{i,j} \lambda_{l,i} (-1)^i \{2i+1\}!/\{1\}. \end{aligned}$$

On the other hand, since  $B = B(H; 1)$ , we have

$$\begin{aligned} J_{A;P'_i,P'_j,S_l} &= J_{B(H;1);P'_i,P'_j,S_l} \\ &= \sum_{k \geq 0} x_{i,j}^{(k)} J_{H;P'_k,S_l} \\ &= x_{i,j}^{(l)} \{2l+1\}_{2l}/\{l\}!. \end{aligned}$$

Here, the last identity follows from Lemma 6.1.

Consequently, we have

$$x_{i,j}^{(l)} = \delta_{i,j} (-1)^i \frac{\{2i+1\}! \{l\}!}{\{2l+1\}!} \lambda_{l,i},$$

which completes the proof.  $\square$

In what follows, we prove Proposition 6.2. We use two more lemmas as follows.

**Lemma 6.3.** *For  $l \geq 0$ , we have*

$$S_l = \sum_{k=0}^l (-1)^k \begin{bmatrix} 2l+1 \\ k \end{bmatrix} V_{2l-2k}.$$

*Proof.* We use an induction on  $l$ . For  $m = 0, 1$ , we have

$$\begin{aligned} S_0 &= 1, \\ S_1 &= (V_2 - (q + 1 + q^{-1})) = V_2 - [3]. \end{aligned}$$

For  $m \geq 2$ , we have

$$\begin{aligned}
S_m &= S_{m-1}(V_2 - (q^m + 1 + q^{-m})) \\
&= \sum_{k=0}^{m-1} (-1)^k \begin{bmatrix} 2m-1 \\ k \end{bmatrix} V_{2m-2k-2} (V_2 - (q^m + 1 + q^{-m})) \\
&= \sum_{k=0}^{m-2} (-1)^k \begin{bmatrix} 2m-1 \\ k \end{bmatrix} (V_{2m-2k} + V_{2m-2k-2} + V_{2m-2k-4} - (q^m + 1 + q^{-m})V_{2m-2k-2}) \\
&\quad + (-1)^{m-1} \begin{bmatrix} 2m-1 \\ m-1 \end{bmatrix} (V_2 - (q^m + 1 + q^{-m})) \\
&= \sum_{k=0}^{m-2} (-1)^k \begin{bmatrix} 2m-1 \\ k \end{bmatrix} (V_{2m-2k} - (q^m + q^{-m})V_{2m-2k-2} + V_{2m-2k-4}) \\
&\quad + (-1)^{m-1} \begin{bmatrix} 2m-1 \\ m-1 \end{bmatrix} (V_2 - (q^m + 1 + q^{-m})) \\
&= V_{2m} - (q^m + q^{-m})V_{2m-2} - [2m-1]V_{2m-2} \\
&\quad + \sum_{k=2}^{m-1} \left( (-1)^{k-2} \begin{bmatrix} 2m-1 \\ k-2 \end{bmatrix} - (-1)^{k-1} \begin{bmatrix} 2m-1 \\ k-1 \end{bmatrix} (q^m + q^{-m}) + (-1)^k \begin{bmatrix} 2m-1 \\ k \end{bmatrix} \right) V_{2m-2k} \\
&\quad + (-1)^{m-2} \begin{bmatrix} 2m-1 \\ m-2 \end{bmatrix} - (-1)^{m-1} \begin{bmatrix} 2m-1 \\ m-1 \end{bmatrix} (q^m + 1 + q^{-m}) \\
&= \sum_{k=1}^m (-1)^k \begin{bmatrix} 2m+1 \\ k \end{bmatrix} V_{2m-2k}.
\end{aligned}$$

□

The following lemma is observed in the proof of [10, Proposition 6.6].

**Lemma 6.4** (Habiro [10]). *For  $m, n \geq 0$ , we have*

$$J_{H;V_m,S_n} = \{m+n+1\}_{2n+1}/\{1\}.$$

Now, we prove Proposition 6.2.

*Proof of Proposition 6.2.* For  $m, n \geq 0$ , we have

$$\begin{aligned}
J_{H;S_m,S_n} &= \sum_{l \geq 0} \lambda_{m,l} \{l\}! J_{H;P'_l,S_n} \\
&= \lambda_{m,n} \{n\}! (\{2n+1\}_{2n}/\{n\}!) \\
&= \lambda_{m,n} \{2n+1\}_{2n}.
\end{aligned} \tag{8}$$

Hence we have

$$\begin{aligned}
\lambda_{m,n} &= J_{H;S_m,S_n}/\{2n+1\}_{2n} \\
&= \sum_{k=1}^m (-1)^k \begin{bmatrix} 2m+1 \\ k \end{bmatrix} J_{H;V_{2m-2k},S_n}/\{2n+1\}_{2n} \\
&= \sum_{k=1}^m (-1)^k \begin{bmatrix} 2m+1 \\ k \end{bmatrix} \{2m+n-2k+1\}_{2n+1}/\{2n+1\}! \\
&= \sum_{k=1}^m (-1)^k \begin{bmatrix} 2m+1 \\ k \end{bmatrix} \begin{bmatrix} 2m+n-2k+1 \\ 2n+1 \end{bmatrix},
\end{aligned}$$

where the second identity follows from Lemma 6.3 and the third identity follows from Lemma 6.4. Hence we have the assertion.  $\square$

We prove Lemma 4.1.

*Proof of Lemma 4.1.* By (8) and symmetric property of  $J_{H;S_m,S_n}$ , we have

$$\begin{aligned}
J_{H;S_m,S_n} &= \lambda_{m,n}\{2n+1\}_{2n} \\
&= \lambda_{n,m}\{2m+1\}_{2m}
\end{aligned}$$

for  $m, n \geq 0$ . Thus we have

$$\begin{aligned}
\lambda_{m,n} &= \frac{\{2m+1\}_{2m}}{\{2n+1\}_{2n}} \lambda_{n,m} \\
&= \frac{\{2m+1\}!}{\{2n+1\}!} \lambda_{n,m}.
\end{aligned}$$

Hence we have the assertion.  $\square$

## 6.2 Proof of Theorem 3.3

We reduce Theorem 3.3 to Proposition 6.5 as follows.

**Proposition 6.5.** *Let  $L = L_1 \cup \cdots \cup L_n$  be an algebraically-split link with in  $S^3$  and set  $\tilde{L} = L_2 \cup L_3 \cup \cdots \cup L_n$ . For  $\epsilon, \epsilon', \epsilon_2, \dots, \epsilon_n \in \{\pm 1\}$ , we have*

$$J_{S^3(B(L;1);\epsilon,\epsilon',\epsilon_2,\dots,\epsilon_n)} - J_{S^3(\tilde{L};\epsilon_2,\dots,\epsilon_n)} \in \Phi_1^2 \Phi_2^2 \Phi_3 \Phi_4 \Phi_6 \widehat{\mathbb{Z}[q]}.$$

*Proof of Theorem 3.3 assuming Proposition 6.5.* Let  $M$  be an integral homology sphere and  $K$  a knot with 0-framing in  $M$ . It is known that there is an algebraically split link  $T = T_1 \cup \cdots \cup T_n$  in  $S^3$  such that  $S^3(T; \epsilon_1, \dots, \epsilon_n)$  is homeomorphic to  $M$ . Here, by isotopy of  $K$  in  $M$ , we can assume that  $K$  is null-homologous in  $M \setminus N$ , where  $N$  is the union of solid tori on which the surgery operation along  $T$  was done. Now,  $K$  may be regarded as a 0-framed knot in  $S^3 \setminus T$ . Set  $L = K \cup T$ . By Proposition 6.5, we have

$$\begin{aligned}
J_{M(B(K);\epsilon,\epsilon')} - J_M &= J_{S^3(B(L;1);\epsilon,\epsilon',\epsilon_1,\dots,\epsilon_n)} - J_{S^3(\tilde{L}=T;\epsilon_1,\dots,\epsilon_n)} \\
&\in \Phi_1^2 \Phi_2^2 \Phi_3 \Phi_4 \Phi_6 \widehat{\mathbb{Z}[q]}.
\end{aligned}$$

Hence we have the assertion.  $\square$



We prove Proposition 6.5. We use the following proposition.

**Lemma 6.6** (Habiro [10, Theorem 8.2]). *Let  $L$  be an  $n$ -component, algebraically-split link with 0-framing. For  $l_1, \dots, l_n \geq 0$ , we have*

$$J_{L;P'_{l_1}, \dots, P'_{l_n}} \in \frac{\{2l_j + 1\}_{l_j+1}}{\{1\}} \mathbb{Z}[q^{1/2}, q^{-1/2}],$$

where  $j$  is an integer such that  $l_j = \max\{l_i\}_{1 \leq i \leq n}$ .

*Proof of Proposition 6.5.* Let  $L$  be an  $n$ -component, algebraically-split link with 0-framing. By the definition, we have

$$\begin{aligned} J_{M(B(L;1); \epsilon, \epsilon', \epsilon_2, \dots, \epsilon_n)} &= J_{B(L;1); \omega^{-\epsilon}, \omega^{-\epsilon'}, \omega^{-\epsilon_2}, \dots, \omega^{-\epsilon_n}} \\ &= \sum_{i \geq 0} \sum_{j \geq 0} (-\epsilon)^i (-\epsilon')^j q^{-\epsilon i(i+3)/4 - \epsilon' j(j+3)/4} J_{B(L;1); P'_i, P'_j, \omega^{-\epsilon_2}, \dots, \omega^{-\epsilon_n}}. \end{aligned}$$

By Lemma 3.1 and Theorem 3.2, we have

$$J_{B(L;1); P'_i, P'_j, \omega^{-\epsilon_2}, \dots, \omega^{-\epsilon_n}} = \delta_{i,j} \sum_{l \geq 0} x_{i,i}^{(l)} J_{L; P'_l, \omega^{-\epsilon_2}, \dots, \omega^{-\epsilon_n}},$$

thus we have

$$\begin{aligned} &\sum_{i \geq 0} \sum_{j \geq 0} (-\epsilon)^i (-\epsilon')^j q^{-\epsilon i(i+3)/4 - \epsilon' j(j+3)/4} J_{B(L;1); P'_i, P'_j, \omega^{-\epsilon_2}, \dots, \omega^{-\epsilon_n}} \\ &= \sum_{i \geq 0} (\epsilon \epsilon')^i q^{-(\epsilon + \epsilon') i(i+3)/4} \sum_{l \geq 0} x_{i,i}^{(l)} J_{L; P'_l, \omega^{-\epsilon_2}, \dots, \omega^{-\epsilon_n}} \\ &= \sum_{l \geq 0} \left( \sum_{i \geq 0} (\epsilon \epsilon')^i q^{-(\epsilon + \epsilon') i(i+3)/4} x_{i,i}^{(l)} \right) J_{L; P'_l, \omega^{-\epsilon_2}, \dots, \omega^{-\epsilon_n}} \\ &= 1 + \sum_{l \geq 1} \left( \sum_{i \geq 0} (\epsilon \epsilon')^i q^{-(\epsilon + \epsilon') i(i+3)/4} x_{i,i}^{(l)} \right) J_{L; P'_l, \omega^{-\epsilon_2}, \dots, \omega^{-\epsilon_n}}. \end{aligned}$$

For  $l \geq 1$ , set

$$\begin{aligned} s_l^{(\epsilon, \epsilon')} &= \sum_{i \geq 0} (\epsilon \epsilon')^i q^{-(\epsilon + \epsilon') i(i+3)/4} x_{i,i}^{(l)} \\ &= \sum_{i \geq \lceil l/2 \rceil}^{2l} (\epsilon \epsilon')^i q^{-(\epsilon + \epsilon') i(i+3)/4} x_{i,i}^{(l)}, \end{aligned}$$

where the second identity follows from Corollary 4.4 (i). It is enough to prove

$$s_l^{(\epsilon, \epsilon')} J_{L; P'_l, P'_{l_2}, \dots, P'_{l_n}} \in \tilde{\Phi}_1^2 \tilde{\Phi}_2^2 \tilde{\Phi}_3 \tilde{\Phi}_4 \tilde{\Phi}_6 \mathbb{Z}[q^{1/2}, q^{-1/2}],$$

for  $l \geq 1$  and  $l_2, \dots, l_n \geq 0$ .

By Lemma 6.6, we have

$$J_{L;P'_1,P'_{l_2},\dots,P'_{l_n}} \in \frac{\{3\}_2}{\{1\}} \mathbb{Z}[q^{1/2}, q^{-1/2}] \subset \tilde{\Phi}_1 \tilde{\Phi}_2 \tilde{\Phi}_3 \mathbb{Z}[q^{1/2}, q^{-1/2}],$$

$$J_{L;P'_2,P'_{l_2},\dots,P'_{l_n}} \in \frac{\{5\}_3}{\{1\}} \mathbb{Z}[q^{1/2}, q^{-1/2}] \subset \tilde{\Phi}_1^2 \tilde{\Phi}_2 \tilde{\Phi}_3 \tilde{\Phi}_4 \tilde{\Phi}_5 \mathbb{Z}[q^{1/2}, q^{-1/2}],$$

and for  $l \geq 3$ , we have

$$J_{L;P'_l,P'_{l_2},\dots,P'_{l_n}} \in \frac{\{2l+1\}_{l+1}}{\{1\}} \mathbb{Z}[q, q^{-1}]$$

$$\subset \tilde{\Phi}_1^2 \tilde{\Phi}_2^2 \tilde{\Phi}_3 \tilde{\Phi}_4 \tilde{\Phi}_6 \mathbb{Z}[q^{1/2}, q^{-1/2}].$$

Thus we have only to prove

$$s_1^{(\epsilon, \epsilon')} \in \tilde{\Phi}_1 \tilde{\Phi}_2 \tilde{\Phi}_4 \tilde{\Phi}_6 \mathbb{Z}[q^{1/2}, q^{-1/2}], \quad (9)$$

$$s_2^{(\epsilon, \epsilon')} \in \tilde{\Phi}_2 \tilde{\Phi}_6 \mathbb{Z}[q^{1/2}, q^{-1/2}]. \quad (10)$$

By Corollaries 4.2 and 4.4, we have

$$x_{1,1}^{(1)} = -\{4\} = -\tilde{\Phi}_1 \tilde{\Phi}_2 \tilde{\Phi}_4,$$

$$x_{2,2}^{(1)} = (-1)^3 \frac{\{5\}!\{1\}!}{\{3\}!\{2\}!} x_{1,1}^{(2)} = ((-1)^3 \frac{\{5\}!\{1\}!}{\{3\}!\{2\}!}) (-\{2\}!) = \tilde{\Phi}_1^3 \tilde{\Phi}_2 \tilde{\Phi}_4 \tilde{\Phi}_5,$$

$$x_{1,1}^{(2)} = -\{2\}! = -\tilde{\Phi}_1^2 \tilde{\Phi}_2,$$

$$x_{2,2}^{(2)} = \tilde{\Phi}_1^2 \tilde{\Phi}_2 (q^{-5} + q^{-4} + 2q^{-3} + q^{-2} + 2q^{-1} + 2 + 2q + q^2 + 2q^3 + q^4 + q^5),$$

$$x_{2,2}^{(3)} = (-1)^2 \{3\}!\{8\}/\{1\} = \tilde{\Phi}_1^3 \tilde{\Phi}_2^2 \tilde{\Phi}_3 \tilde{\Phi}_4 \tilde{\Phi}_8,$$

$$x_{2,2}^{(4)} = \{4\}! = \tilde{\Phi}_1^4 \tilde{\Phi}_2^2 \tilde{\Phi}_3 \tilde{\Phi}_4.$$

Thus we have

$$\begin{aligned}
s_1^{(-1,-1)} &= \sum_{i=1}^2 q^{i(i+3)/2} x_{i,i}^{(1)} \\
&= \Phi_1 \Phi_2 \Phi_4 \Phi_6 \cdot q(1 - q + q^3), \\
s_2^{(-1,-1)} &= \sum_{i=1}^4 q^{i(i+3)/2} x_{i,i}^{(2)} \\
&= \Phi_1^2 \Phi_2 \Phi_3 \Phi_6 \cdot q^{5/2} (1 + q + q^2 + q^3 + q^4 + q^5 + q^6 - q^{13} - q^{14} - q^{16} + q^{21}), \\
s_1^{(1,1)} &= \sum_{i=1}^2 q^{-i(i+3)/2} x_{i,i}^{(1)} \\
&= -\Phi_1 \Phi_2 \Phi_4 \Phi_6 \cdot q^{-10} (-1 + q^2 + q^3), \\
s_2^{(1,1)} &= \sum_{i=1}^4 q^{-i(i+3)/2} x_{i,i}^{(2)} \\
&= \Phi_1^2 \Phi_2 \Phi_3 \Phi_6 \cdot q^{-61/2} (1 - q^2 - q^7 - q^8 + q^{15} + q^{16} + q^{17} + q^{18} + q^{19} + q^{20} + q^{21}), \\
s_1^{(-1,1)} &= s_1^{(1,-1)} = \sum_{i=1}^2 (-1)^i x_{i,i}^{(1)} \\
&= \Phi_1 \Phi_2 \Phi_4 \Phi_6 \Phi_{12} \cdot q^{-5}, \\
s_2^{(-1,1)} &= s_2^{(1,-1)} = \sum_{n=1}^4 (-1)^n x_{n,n}^{(2)} \\
&= \Phi_1^2 \Phi_2 \Phi_6 \cdot q^{-33/2} (1 + q - q^3 - q^4 + q^5 + 2q^6 + q^7 - q^8 - 2q^9 + q^{10} + 4q^{11} + 4q^{12} \\
&\quad - 3q^{14} + 4q^{16} + 4q^{17} + q^{18} - 2q^{19} - q^{20} + q^{21} + 2q^{22} + q^{23} - q^{24} - q^{25} + q^{27} + q^{28}).
\end{aligned}$$

Hence we have (9) and (10). This completes the proof.  $\square$

### 6.3 Proof of Proposition 3.7

To prove Proposition 3.7, we use the following lemma.

**Lemma 6.7** (Habiro[10, Proposition 14.5]). *For  $i, j, k \in \mathbb{Z}$ , we have*

$$J_{M_{i,j,k}} = \sum_{l \geq 0} \omega_{i,l} \omega_{j,l} \omega_{k,l} (-1)^l \{2l+1\}_{l+1} / \{1\},$$

where for  $p \in \mathbb{Z}$  and  $n \geq 0$ ,

$$\omega_{p,n} = \begin{cases} q^{\frac{1}{4}n(n+3)} \sum_{\mathbf{i} \in S(n,p)} \begin{bmatrix} n \\ \mathbf{i} \end{bmatrix}_q q^{f(\mathbf{i})} & \text{for } p \geq 0, \\ (-1)^n q^{-\frac{1}{4}n(n+3)} \sum_{\mathbf{i} \in S(n,-p)} \begin{bmatrix} n \\ \mathbf{i} \end{bmatrix}_{q^{-1}} q^{-f(\mathbf{i})} & \text{for } p \leq 0. \end{cases}$$

Here for  $p, n \geq 0$ , we set

$$S(n, p) = \{(i_1, \dots, i_p) \mid i_1, \dots, i_p \geq 0, i_1 + \dots + i_p = n\},$$

and for  $\mathbf{i} = (i_1, \dots, i_p) \in S(n, p)$ , we set

$$\begin{bmatrix} n \\ \mathbf{i} \end{bmatrix}_q = \frac{[n]_q!}{[i_1]_q! \cdots [i_p]_q!}, \quad f(\mathbf{i}) = \sum_{j=1}^{p-1} (s_j^2 + s_j),$$

where  $s_j = \sum_{k=1}^j i_k$  and

$$[m]_q = \frac{q^m - 1}{q - 1}, \quad [m]_q! = [m]_q [m-1]_q \cdots [1]_q,$$

for  $m \geq 0$ .

*Proof of Proposition 3.7.* By Lemma 6.7 and  $\omega_{p,0} = 1$  for  $p \in \mathbb{Z}$ , we have

$$\begin{aligned} J_{M_{i,j,k}} - 1 &= \sum_{l \geq 1} \omega_{i,l} \omega_{j,l} \omega_{k,l} (-1)^l \{2l+1\}_{l+1} / \{1\} \\ &\equiv -\omega_{i,1} \omega_{j,1} \omega_{k,1} \tilde{\Phi}_3 \tilde{\Phi}_2 \tilde{\Phi}_1 + \omega_{i,2} \omega_{j,2} \omega_{k,2} \tilde{\Phi}_5 \tilde{\Phi}_4 \tilde{\Phi}_3 \tilde{\Phi}_2 \tilde{\Phi}_1^2 \pmod{\tilde{\Phi}_2^2} \\ &\equiv -q^{-2} \omega_{i,1} \omega_{j,1} \omega_{k,1} \Phi_3 \Phi_2 \Phi_1 + q^{-5-\frac{1}{2}} \omega_{i,2} \omega_{j,2} \omega_{k,2} \Phi_5 \Phi_4 \Phi_3 \Phi_2 \Phi_1^2 \pmod{\Phi_2^2}. \end{aligned}$$

Thus we have

$$\frac{J_{M_{i,j,k}} - 1}{\Phi_2} = -q^{-2} \omega_{i,1} \omega_{j,1} \omega_{k,1} \Phi_3 \Phi_1 + q^{-5-\frac{1}{2}} \omega_{i,2} \omega_{j,2} \omega_{k,2} \Phi_5 \Phi_4 \Phi_3 \Phi_1^2 \pmod{\Phi_2}.$$

For  $p \geq 0$ , we have

$$\begin{aligned} \omega_{p,1} &= q^{2p+1} \sum_{t=1}^p q^{-2t}, \\ \omega_{-p,1} &= -q^{-2p-1} \sum_{t=1}^p q^{2t}, \\ \omega_{p,2} &= q^{\frac{5}{2}} \left( \sum_{t=1}^p q^{6(p-t)} + [2]_q \sum_{1 \leq s < t \leq p} q^{2(t-s+1)+6(p-t)} \right), \\ \omega_{-p,2} &= q^{-\frac{5}{2}} \left( \sum_{t=1}^p q^{-6(p-t)} + [2]_{q^{-1}} \sum_{1 \leq s < t \leq p} q^{-2(t-s+1)-6(p-t)} \right). \end{aligned}$$

Thus, for  $p \in \mathbb{Z}$ , we have

$$\omega_{p,1}|_{q=-1} = -p, \quad q^{-\frac{1}{2}} \omega_{p,2}|_{q=-1} = p.$$

Together with

$$\Phi_1|_{q=-1} = -2, \quad \Phi_3|_{q=-1} = 1, \quad \Phi_4|_{q=-1} = 2, \quad \Phi_5|_{q=-1} = 1,$$

we have

$$\begin{aligned} -q^{-2}\omega_{i,1}\omega_{j,1}\omega_{k,1}\Phi_3\Phi_1|_{q=-1} &= -2ijk, \\ q^{-5-\frac{1}{2}}\omega_{i,2}\omega_{j,2}\omega_{k,2}\Phi_5\Phi_4\Phi_3\Phi_1^2|_{q=-1} &= 8ijk. \end{aligned}$$

Hence we have the assertion.  $\square$

**Acknowledgments.** This work was partially supported by JSPS Research Fellowships for Young Scientists. The author is deeply grateful to Professor Kazuo Habiro and Professor Tomotada Ohtsuki for helpful advice and encouragement.

## References

- [1] R. H. Bing, A homeomorphism between the 3-sphere and the sum of two solid horned spheres, *Ann. of Math. (2)* **56** (1952) 354–362.
- [2] D. Cimasoni, Slicing Bing doubles. *Algebr. Geom. Topol.* **6** (2006), 2395–2415.
- [3] J. C. Cha, Link concordance, homology cobordism, and Hirzebruch-type defects from iterated p-covers. *J. Eur. Math. Soc. (JEMS)* **12** (2010), no. 3, 555–610.
- [4] J. C. Cha, T. Kim, Covering link calculus and iterated Bing doubles. *Geom. Topol.* **12** (2008), no. 4, 2173–2201.
- [5] J. C. Cha, C. Livingston, D. Ruberman, Algebraic and Heegaard-Floer invariants of knots with slice Bing doubles. *Math. Proc. Cambridge Philos. Soc.* **144** (2008), no. 2, 403–410.
- [6] T. D. Cochran, Derivatives of links: Milnor’s concordance invariants and Massey’s products. *Mem. Amer. Math. Soc.* 84 (1990), no. 427, x+73 pp.
- [7] T. D. Cochran, S. Harvey, C. Leidy, Link concordance and generalized doubling operators. *Algebr. Geom. Topol.* **8** (2008), no. 3, 1593–1646.
- [8] M. Freedman, X. S. Lin, On the (A,B)-slice problem. *Topology* **28** (1989), no. 1, 91–110.
- [9] S. Harvey, Homology cobordism invariants and the Cochran-Orr-Teichner filtration of the link concordance group. *Geom. Topol.* **12** (2008), no. 1, 387–430.
- [10] K. Habiro, A unified Witten-Reshetikhin-Turaev invariants for integral homology spheres. *Invent. Math.* **171** (2008), no. 1, 1–81.
- [11] K. Habiro, Claspers and finite type invariants of links. *Geom. Topol.* **4** (2000), 1–83.
- [12] H. R. Morton, P. Strickland, Jones polynomial invariants for knots and satellites. *Math. Proc. Cambridge Philos. Soc.* **109** (1991), no. 1, 83–103.

- [13] M. Petkovšek, H. S. Wilf, D. Zeilberger,  $A=B$ . With a foreword by Donald E. Knuth. With a separately available computer disk. A K Peters, Ltd., Wellesley, MA, 1996. xii+212 pp.
- [14] N. Y. Reshetikhin, V. G. Turaev, Ribbon graphs and their invariants derived from quantum groups. *Comm. Math. Phys.* **127** (1990), no. 1, 1–26.